

**$SU(2) \times SU(2)$ harmonic superspace and
(4,4) sigma models with torsion**

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August 1995, Buckow, Germany**Abstract**

We review a manifestly supersymmetric off-shell formulation of a wide class of torsionful (4,4) $2D$ sigma models and their massive deformations in the harmonic superspace with a double set of $SU(2)$ harmonic variables. Sigma models with both commuting and non-commuting left and right complex structures are treated.

1. Introduction. Manifestly supersymmetric off-shell superfield formulations of $2D$ sigma models with extended worldsheet SUSY are most appropriate for revealing remarkable target geometries of these theories. For torsionless (2,2) and (4,4) sigma models the relevant superfield Lagrangians coincide with (or are directly related to) the fundamental objects underlying the bosonic target geometry: Kähler potential in the (2,2) case [1], hyper-Kähler or quaternionic-Kähler potentials in the flat or curved (4,4) cases [2 - 5]. One of the basic advantages of such a description is the possibility to explicitly compute the corresponding bosonic metrics (Kähler, hyper-Kähler, quaternionic ...) starting from an unconstrained superfield action [2, 6]. This is to be compared with the approach when only (1,1) supersymmetry is kept manifest, while the restrictions imposed by higher extended (on-shell) supersymmetries amount to the existence of an appropriate number of covariantly constant complex structures on the bosonic manifold. Such an approach yields no explicit recipes for computing the bosonic metrics. To have superfield off-shell formulations with all supersymmetries manifest is also highly desirable while quantizing these theories and proving their ultraviolet finiteness.

An important class of $2D$ supersymmetric sigma models is given by (2,2) and (4,4) models with torsionful bosonic target manifolds and two independent left and right sets of complex structures (see, e.g. [7, 8]). These models and, in particular, their group manifold WZNW representatives [9] can provide non-trivial backgrounds for $4D$ superstrings (see, e.g., [10]) and be relevant to $2D$ black holes in the stringy context [11, 12]. A manifestly supersymmetric formulation of (2,2) models with commuting left and right complex structures in terms of chiral and twisted chiral (2,2) superfields and an exhaustive discussion of their geometry have been given in [7]. For (4,4) models with commuting structures there exist manifestly supersymmetric off-shell formulations in the projective, ordinary

and $SU(2) \times SU(2)$ harmonic $(4, 4)$ superspaces [11,13,14]. The appropriate superfields represent, in one or another way, the $(4, 4)$ $2D$ twisted multiplet [15, 7].

Much less is known about $(2, 2)$ and $(4, 4)$ sigma models with non-commuting complex structures, including most of group manifold ones [9]. In particular, it is unclear how to describe them off shell in general. As was argued in Refs. [7, 16, 11], twisted $(2, 2)$ and $(4, 4)$ multiplets are not suitable for this purpose. It has been then suggested to make use of some other off-shell representations of $(2, 2)$ [16, 17] and $(4, 4)$ [16, 18] worldsheet SUSY. However, it is an open question whether the relevant actions correspond to generic sigma models of this type.

In this talk we review another approach to the off-shell description of general $(4, 4)$ sigma models with torsion and their massive deformations, largely exploiting an analogy with general torsionless hyper-Kähler $(4, 4)$ sigma models in $SU(2)$ harmonic superspace [2 - 4]. The basic tool is the new type of harmonic superspace containing two independent sets of harmonic variables, the $SU(2) \times SU(2)$ harmonic superspace. We demonstrate that it allows to construct off-shell formulations for a wide class of torsionful $(4, 4)$ sigma models with commuting as well as non-commuting left and right quaternionic structures [14, 19, 20].

2. $(4,4)$ twisted multiplet in $SU(2) \times SU(2)$ harmonic superspace. The $SU(2) \times SU(2)$ harmonic superspace is an extension of the standard real $(4, 4)$ $2D$ superspace by two independent sets of harmonic variables $u^{\pm 1 i}$ and $v^{\pm 1 a}$ ($u^{1 i} u_i^{-1} = v^{1 a} v_a^{-1} = 1$) associated with the automorphism groups $SU(2)_L$ and $SU(2)_R$ of the left and right sectors of $(4, 4)$ supersymmetry [14]. The corresponding analytic subspace is spanned by the following set of coordinates

$$(\zeta, u, v) = (x^{++}, x^{--}, \theta^{1,0 \underline{i}}, \theta^{0,1 \underline{a}}, u^{\pm 1 i}, v^{\pm 1 a}) , \quad (1)$$

where we omitted the light-cone indices of odd coordinates (the first and second θ s in (1) carry, respectively, the indices $+$ and $-$). The superscript “ n, m ” stands for two independent harmonic $U(1)$ charges, left (n) and right (m) ones. The additional doublet indices of odd coordinates, \underline{i} and \underline{a} , refer to two extra automorphism groups $SU(2)'_L$ and $SU(2)'_R$ which together with $SU(2)_L$ and $SU(2)_R$ form the full $(4, 4)$ $2D$ supersymmetry automorphism group $SO(4)_L \times SO(4)_R$.

It was argued in [14] that this type of harmonic superspace is most appropriate for constructing off-shell formulations of general $(4, 4)$ sigma models with torsion. This hope mainly relied upon the fact that the twisted $(4, 4)$ multiplet has a natural description as a real analytic $SU(2) \times SU(2)$ harmonic superfield $q^{1,1}(\zeta, u, v)$ subjected to the harmonic constraints

$$D^{2,0} q^{1,1} = D^{0,2} q^{1,1} = 0 . \quad (2)$$

where

$$\begin{aligned} D^{2,0} &= \partial^{2,0} + i\theta^{1,0 \underline{i}} \partial_{\underline{i}}^{1,0} \partial_{++} , \quad D^{0,2} = \partial^{0,2} + i\theta^{0,1 \underline{a}} \partial_{\underline{a}}^{0,1} \partial_{--} \\ (\partial^{2,0} &= u^{1 i} \frac{\partial}{\partial u^{-1 i}} , \quad \partial^{0,2} = v^{1 a} \frac{\partial}{\partial v^{-1 a}}) \end{aligned} \quad (3)$$

are the left and right mutually commuting analyticity-preserving harmonic derivatives. These constraints leave in $q^{1,1}$ $8 + 8$ independent components that is just the irreducible

off-shell component content of $(4, 4)$ twisted multiplet. The most general off-shell action of n such multiplets $q^{1,1 M} (M = 1, \dots, n)$ is given by the following integral over the analytic superspace (1)

$$S_q = \int \mu^{-2,-2} h^{2,2}(q^{1,1 M}, u^{\pm 1}, v^{\pm 1}) , \quad (4)$$

$\mu^{-2,-2}$ being the relevant integration measure. The analytic superfield lagrangian $h^{2,2}$ is an arbitrary function of its arguments (consistent with the external $U(1)$ harmonic charges 2, 2).

Let us write down the physical bosons part of the action (4)

$$S_{phb} = \frac{1}{2} \int d^2 z \{ G_{Mia Njb}(q) \partial_{++} q^{ia M} \partial_{--} q^{jb N} + B_{Mia Njb}(q) \partial_{++} q^{ia M} \partial_{--} q^{jb N} \} . \quad (5)$$

Here

$$G_{Mia Njb}(q) = G_{MN}(q) \epsilon_{ij} \epsilon_{ab} , \quad (6)$$

$$B_{Mia Njb}(q) = \int dudv g_{MN}(q_0^{1,1}, u, v) [\epsilon_{ij} v_a^1 v_b^{-1} - \epsilon_{ab} u_i^1 u_j^{-1}] , \quad (7)$$

$$G_{MN}(q) = \int dudv g_{MN}(q_0^{1,1}, u, v) , \quad g_{MN}(q_0^{1,1}, u, v) = \frac{\partial^2 h^{2,2}}{\partial q^{1,1 M} \partial q^{1,1 N}}|_{\theta=0} , \quad (8)$$

where $q_0^{1,1} \equiv q^{1,1}|_{\theta=0}$. The symmetric and skew-symmetric objects $G_{Mia Njb}$ and $B_{Mia Njb}$ can be identified with the metric and torsion potential on the target space.

It is advantageous to rewrite the second term in (5) through the torsion field strength:

$$H_{Mia Njb Tkd} = \partial_{Mia} B_{Njb Tkd} + \partial_{Tkd} B_{Mia Njb} + \partial_{Njb} B_{Tkd Mia} , \quad (9)$$

where $\partial_{Mid} \equiv \partial / \partial q^{id M}$. Letting $q^{ia M}$ depend on an extra parameter t , with $q^{ia M}(t)|_{t=1} \equiv q^{ia M}$, $q^{ia M}(t)|_{t=0} = \epsilon^{ia}$, one can locally rewrite the torsion term as

$$B_{Mia Njb} \partial_{++} q^{ia M} \partial_{--} q^{jb N} = \int_0^1 dt H_{Mia Njb Tkd} \partial_t q^{ia M} \partial_{++} q^{jb N} \partial_{--} q^{kd T} . \quad (10)$$

For $B_{Mia Njb}$ given by eq. (7), $H_{Mia Njb Tkd}$ is reduced to

$$H_{Mia Njb Tkd}(q) = \partial_{(Mid} G_{N T)}(q) \epsilon_{ab} \epsilon_{jk} + \partial_{(Mka} G_{N T)}(q) \epsilon_{db} \epsilon_{ij} . \quad (11)$$

Note that all the fermionic terms in the action (4) are also expressed through the same function $G_{MN}(q)$ and its derivatives.

In the case of four-dimensional targets (the case of one $q^{1,1}$) the metric, as is seen from eqs. (6) - (8), is reduced to a conformal factor

$$G_{ia jb}(q) = \epsilon_{ij} \epsilon_{ab} G(q) \equiv \epsilon_{ij} \epsilon_{ab} \int dudv g(q_0^{1,1}, u, v) , \quad g = \frac{\partial^2}{\partial q_0^{1,1} \partial q_0^{1,1}} h^{2,2}(q_0^{1,1}, u, v) \quad (12)$$

which satisfies the Laplace equation

$$\square G(q) \equiv \partial^{ia} \partial_{ia} G(q) = 2 \int dudv \left(\frac{\partial^2}{\partial q_0^{-1,-1} \partial q_0^{1,1}} - \frac{\partial^2}{\partial q_0^{-1,1} \partial q_0^{1,-1}} \right) g(q_0^{1,1}, u, v) = 0 . \quad (13)$$

This agrees with the general conditions on the bosonic target in the torsionful $(4, 4)$ sigma models with four-dimensional targets [7, 12].

As a non-trivial example of the $q^{1,1}$ action with four-dimensional bosonic manifold we give the action of $(4, 4)$ extension of the $SU(2) \times U(1)$ WZNW sigma model

$$S_{wzw} = \frac{1}{4\kappa^2} \int \mu^{-2,-2} \hat{q}^{1,1} \hat{q}^{(1,1)} \left(\frac{\ln(1+X)}{X^2} - \frac{1}{(1+X)X} \right). \quad (14)$$

Here

$$\hat{q}^{1,1} = q^{1,1} - c^{1,1}, \quad X = c^{-1,-1} \hat{q}^{1,1}, \quad c^{\pm 1, \pm 1} = c^{ia} u_i^{\pm 1} v_a^{\pm 1}, \quad c^{ia} c_{ia} = 2. \quad (15)$$

Despite the presence of an extra quartet constant c^{ia} in the analytic superfield lagrangian, the action (14) actually does not depend on c^{ia} [14] as it is invariant under arbitrary rigid rescalings and $SU(2) \times SU(2)$ rotations of this constant. Its physical bosons part is given by the general expression (5) and eventually turns out to be expressed through the single function $G(q)$ which in the case under consideration reads, up to the overall coupling constant,

$$G(q) = \int dudv \frac{1-X}{(1+X)^3} = 2(q^{ia} q_{ia})^{-1} = e^{-2u}, \quad (16)$$

where we have introduced the singlet scalar field $u(x)$ through the polar decomposition

$$q^{ia} = e^{u(z)} \tilde{q}^{ia}(z), \quad \tilde{q}^{ia} \tilde{q}_a^j = \epsilon^{ji}, \quad \tilde{q}^{ia} \tilde{q}_i^b = \epsilon^{ba}. \quad (17)$$

In this parametrization, the resulting action of physical bosons is that of the $SU(2) \times U(1)$ WZNW sigma model

$$\begin{aligned} S_{wzw}^{bos} &= \frac{1}{4\kappa^2} \int d^2z \{ \partial_{++} u \partial_{--} u + \frac{1}{2} \partial_{++} \tilde{q}^{ia} \partial_{--} \tilde{q}_{ia} \\ &\quad + \frac{1}{2} \int_0^1 dt \partial_t \tilde{q}_{ia} \tilde{q}_{jb} (\partial_{++} \tilde{q}^{ib} \partial_{--} \tilde{q}^{ja} - \partial_{++} \tilde{q}^{ja} \partial_{--} \tilde{q}^{ib}) \}. \end{aligned} \quad (18)$$

The fermionic and auxiliary fields parts of the action (14) also have the appropriate form [14].

The last topic we wish to discuss in connection with the action (4) concerns massive deformations of the latter. Surprisingly, and this is a crucial difference of the considered $(4, 4)$ case from, say, $(2, 2)$ sigma models with torsion, the only massive term of $q^{1,1 M}$ consistent with analyticity and off-shell $(4, 4)$ supersymmetry (not modified by central charges) is the following one [13, 14]

$$S_m = m \int \mu^{-2,-2} \theta^{1,0 \dot{i}} \theta^{0,1 \dot{i}} C_{\dot{i} \dot{b}}^M q^{1,1 M}; \quad [m] = cm^{-1}, \quad (19)$$

where $C_{\dot{i} \dot{b}}^M$ are arbitrary constants (subject to the appropriate reality conditions for the modified action to remain real). It immediately follows that, despite the presence of explicit θ 's, (19) is invariant under rigid $(4, 4)$ supersymmetry: one represents the super-translation of, say, $\theta^{1,0 \dot{i}}$ as

$$\delta_{SUSY} \theta^{1,0 \dot{i}} = \epsilon^{ki} u_k^1 = D^{2,0} \epsilon^{ki} u_k^{-1},$$

integrates by parts with respect to $D^{2,0}$ and makes use of the defining constraints (2).

After adding the term (19) to the action (4), passing to components and eliminating auxiliary fields, the effective addition to the (4,4) sigma model component action is given by

$$S_q^{pot} = \frac{m^2}{2} \int d^2z G^{M N}(q) (C_{\underline{i} \underline{a}}^M C^{\underline{i} \underline{a} N}), \quad (20)$$

where $G^{M N}(q)$ is the inverse of the metric $G_{M N}(q)$ defined in eq. (8). Thus we see that the potential term in the case in question is uniquely determined by the form of the bosonic target metric. In particular, in the case of (4,4) $SU(2) \times U(1)$ WZNW model one gets the Liouville potential term for the field $u(x)$, so the massive deformation of this model is nothing but the (4,4) super Liouville theory [15]. It would be interesting to inquire whether (4,4) extensions of other integrable 2D theories (e.g., sine-Gordon theory) can be obtained as massive deformations of some appropriate torsionful (4,4) sigma models.

Note that only for non-trivial curved bosonic targets (with non-trivial dependence on $q^{1,1}$ in $G^{M N}$) the above term actually produces a mass for $q^{1,1}$. One cannot gain a mass for $q^{1,1}$ in this way, starting with the free lagrangian $h^{2,2} \sim q^{1,1 M} q^{1,1 M}$ (corresponding to the $U(1)^n$ bosonic target). This becomes possible only after including into the game another type of (4,4) twisted multiplet which has no simple description in the considered type of (4,4) harmonic superspace [13, 21].

3. More general (4,4) sigma models with torsion. The above $SU(2) \times SU(2)$ harmonic superspace description of (4,4) twisted multiplet suggests a new off-shell formulation of the latter via unconstrained analytic superfields. After implementing the constraints (2) in the action with superfield lagrange multipliers and adding this term to (4) we arrive at the following new action [14]

$$S_{q,\omega} = \int \mu^{-2,-2} \{ q^{1,1 M} (D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{1,-1 M}) + h^{2,2}(q^{1,1}, u, v) \}. \quad (21)$$

In (21) all the involved superfields are unconstrained analytic, so from the beginning the action (21) contains an infinite number of auxiliary fields coming from the double harmonic expansions with respect to the harmonics $u^{\pm 1 i}, v^{\pm 1 a}$. Varying with respect to the Lagrange multipliers $\omega^{1,-1 M}, \omega^{-1,1 M}$ takes one back to the action (4) and constraints (2). On the other hand, varying with respect to $q^{1,1 M}$ yields an algebraic equation for the latter which allows one to get a new dual off-shell representation of the twisted multiplet action through unconstrained analytic superfields $\omega^{-1,1 M}, \omega^{1,-1 M}$.

The crucial feature of the action (21) (and its ω representation) is the abelian gauge invariance

$$\delta \omega^{1,-1 M} = D^{2,0} \sigma^{-1,-1 M}, \quad \delta \omega^{-1,1 M} = -D^{0,2} \sigma^{-1,-1 M}, \quad (22)$$

where $\sigma^{-1,-1 M}$ are unconstrained analytic superfield parameters. This gauge freedom ensures the on-shell equivalence of the q, ω or ω formulations of the twisted multiplet action to its original q formulation (4): it neutralizes superfluous physical dimension component fields in the superfields $\omega^{1,-1 M}$ and $\omega^{-1,1 M}$ and thus equalizes the number of propagating fields in both formulations. It holds already at the free level, with $h^{2,2}$ quadratic in $q^{1,1 M}$, so it is natural to expect that any reasonable generalization of the

action (21) respects this symmetry or a generalization of it. We will see that this is indeed so.

The dual twisted multiplet action (21) is a good starting point for constructing more general actions which embrace sigma models with non-commuting left and right complex structures.

As a natural generalization of (21) we insert an arbitrary dependence on $\omega^{-1,1 M}$, $\omega^{1,-1 M}$ into $h^{2,2}$. In other words, as an ansatz for the general action we take the following one

$$S_{gen} = \int \mu^{-2,-2} \{ q^{1,1 M} (D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{1,-1 M}) + H^{2,2}(q^{1,1}, \omega^{1,-1}, \omega^{-1,1}, u, v) \} , \quad (23)$$

where for the moment the ω dependence in $H^{2,2}$ is not fixed. In refs. [19, 20] we have shown that one can arrive at this ansatz proceeding from the most general form of q, ω action.

It turns out that the ω dependence of the potential $H^{2,2}$ in (23) is actually completely specified by the integrability conditions following from the commutativity condition

$$[D^{2,0}, D^{0,2}] = 0 . \quad (24)$$

To show this, we first write the equations of motion corresponding to (23)

$$D^{2,0} \omega^{-1,1 M} + D^{0,2} \omega^{1,-1 M} = - \frac{\partial H^{2,2}(q, \omega, u, v)}{\partial q^{1,1 M}} , \quad (25)$$

$$D^{2,0} q^{1,1 M} = \frac{\partial H^{2,2}(q, \omega, u, v)}{\partial \omega^{-1,1 M}} , \quad D^{0,2} q^{1,1 M} = \frac{\partial H^{2,2}(q, \omega, u, v)}{\partial \omega^{1,-1 M}} . \quad (26)$$

Applying the integrability condition (24) to the pair of equations (26) and making a natural assumption that it is satisfied as a consequence of the equations of motion (i.e. does not give rise to any new dynamical restrictions), after some algebra we arrive at the set of self-consistency conditions [19, 20]. One of their consequences is the following restriction on the ω dependence of $H^{2,2}$

$$\begin{aligned} H^{2,2} &= h^{2,2}(q, u, v) + \omega^{1,-1 N} h^{1,3 N}(q, u, v) + \omega^{-1,1 N} h^{3,1 N}(q, u, v) \\ &\quad + \omega^{-1,1 N} \omega^{1,-1 M} h^{2,2 [N,M]}(q, u, v) . \end{aligned} \quad (27)$$

Plugging this expression back into the self-consistency relations, one finally deduces four independent constraints on the potentials $h^{2,2}$, $h^{1,3 N}$, $h^{3,1 N}$ and $h^{2,2 [N,M]}$

$$\nabla^{2,0} h^{1,3 N} - \nabla^{0,2} h^{3,1 N} + h^{2,2 [N,M]} \frac{\partial h^{2,2}}{\partial q^{1,1 M}} = 0 \quad (28)$$

$$\nabla^{2,0} h^{2,2 [N,M]} - \frac{\partial h^{3,1 N}}{\partial q^{1,1 T}} h^{2,2 [T,M]} + \frac{\partial h^{3,1 M}}{\partial q^{1,1 T}} h^{2,2 [T,N]} = 0 \quad (29)$$

$$\nabla^{0,2} h^{2,2 [N,M]} - \frac{\partial h^{1,3 N}}{\partial q^{1,1 T}} h^{2,2 [T,M]} + \frac{\partial h^{1,3 M}}{\partial q^{1,1 T}} h^{2,2 [T,N]} = 0 \quad (30)$$

$$h^{2,2 [N,T]} \frac{\partial h^{2,2 [M,L]}}{\partial q^{1,1 T}} + h^{2,2 [L,T]} \frac{\partial h^{2,2 [N,M]}}{\partial q^{1,1 T}} + h^{2,2 [M,T]} \frac{\partial h^{2,2 [L,N]}}{\partial q^{1,1 T}} = 0 , \quad (31)$$

$$\nabla^{2,0} = \partial^{2,0} + h^{3,1 N} \frac{\partial}{\partial q^{1,1 N}}, \quad \nabla^{0,2} = \partial^{0,2} + h^{1,3 N} \frac{\partial}{\partial q^{1,1 N}}. \quad (32)$$

Here $\partial^{2,0}, \partial^{0,2}$ act only on the “target” harmonics, i.e. those appearing explicitly in the potentials.

Thus the true analog of the generic hyper-Kähler (4, 4) sigma model action [2 - 4] in the torsionful case is the action

$$S_{q,\omega} = \int \mu^{-2,-2} \{ q^{1,1 M} D^{0,2} \omega^{1,-1 M} + q^{1,1 M} D^{2,0} \omega^{-1,1 M} + \omega^{1,-1 M} h^{1,3 M} + \omega^{-1,1 M} h^{3,1 M} + \omega^{-1,1 M} \omega^{1,-1 N} h^{2,2 [M,N]} + h^{2,2} \}, \quad (33)$$

where the involved potentials depend only on $q^{1,1 M}$ and target harmonics and satisfy the target space constraints (28) - (31). To reveal the geometry hidden in these constraints we need their general solution, which is still unknown. At present we are only aware of some particular solution which will be presented below.

The action (33) and constraints (28) - (31) enjoy a set of invariances.

One of them is a mixture of reparametrizations in the target space (spanned by the involved superfields and target harmonics) and the transformations which are bi-harmonic analogs of hyper-Kähler transformations of Refs. [23, 4].

More interesting is another invariance which has no analog in the hyper-Kähler case and is a non-abelian and in general nonlinear generalization of the abelian gauge invariance (22)

$$\begin{aligned} \delta \omega^{1,-1 M} &= \left(D^{2,0} \delta^{MN} + \frac{\partial h^{3,1 N}}{\partial q^{1,1 M}} \right) \sigma^{-1,-1 N} - \omega^{1,-1 L} \frac{\partial h^{2,2 [L,N]}}{\partial q^{1,1 M}} \sigma^{-1,-1 N}, \\ \delta \omega^{-1,1 M} &= - \left(D^{0,2} \delta^{MN} + \frac{\partial h^{1,3 N}}{\partial q^{1,1 M}} \right) \sigma^{-1,-1 N} - \omega^{-1,1 L} \frac{\partial h^{2,2 [L,N]}}{\partial q^{1,1 M}} \sigma^{-1,-1 N}, \\ \delta q^{1,1 M} &= \sigma^{-1,-1 N} h^{2,2 [N,M]}. \end{aligned} \quad (34)$$

As expected, the action is invariant only with taking account of the integrability conditions (28) - (31). In general, these gauge transformations close with a field-dependent Lie bracket parameter:

$$\delta_{br} q^{1,1 M} = \sigma_{br}^{-1,-1 N} h^{2,2 [N,M]}, \quad \sigma_{br}^{-1,-1 N} = -\sigma_1^{-1,-1 L} \sigma_2^{-1,-1 T} \frac{\partial h^{2,2 [L,T]}}{\partial q^{1,1 N}}. \quad (35)$$

We see that eq. (31) guarantees the nonlinear closure of the algebra of gauge transformations (34) and so it is a group condition similar to the Jacobi identity.

Curiously enough, the gauge transformations (34) augmented with the group condition (31) are precise bi-harmonic counterparts of the two-dimensional version of basic relations of the so called Poisson nonlinear gauge theory which recently received some attention [22]. The manifold (q, u, v) can be interpreted as a kind of bi-harmonic extension of some Poisson manifold and the potential $h^{2,2 [N,M]}(q, u, v)$ as a tensor field inducing the Poisson structure on this extension.

It should be pointed out that it is the presence of the antisymmetric potential $h^{2,2 [N,M]}$ that makes the considered case non-trivial and, in particular, the gauge invariance (34)

non-abelian. If $h^{2,2[N,M]}$ is vanishing, the invariance gets abelian and the constraints (29) - (31) are identically satisfied, while (28) is solved by

$$h^{1,3M} = \nabla^{0,2}\Sigma^{1,1M}(q, u, v), \quad h^{3,1M} = \nabla^{2,0}\Sigma^{1,1M}(q, u, v), \quad (36)$$

with $\Sigma^{1,1M}$ being an unconstrained prepotential. Then, using the freedom with respect to the target space reparametrizations, one may entirely gauge away $h^{1,3M}, h^{3,1M}$, thereby reducing (33) to the dual action of twisted (4, 4) multiplets (21). In the case of one triple $q^{1,1}, \omega^{1,-1}, \omega^{-1,1}$ the potential $h^{2,2[N,M]}$ vanishes identically, so the general action (23) for $n = 1$ is actually equivalent to (21). Thus *only for* $n \geq 2$ a new class of torsionful (4, 4) sigma models comes out. It is easy to see that the action (33) with non-zero $h^{2,2[N,M]}$ *does not* admit any duality transformation to the form with the superfields $q^{1,1M}$ only, because it is impossible to remove the dependence on $\omega^{1,-1N}, \omega^{-1,1N}$ from the equations for $q^{1,1M}$ by any local field redefinition with preserving harmonic analyticity. Moreover, in contradistinction to the constraints (2), these equations are compatible only with using the equation for ω 's. So, the obtained system certainly does not admit in general any dual description in terms of twisted (4, 4) superfields. Hence, the left and right complex structures on the target space can be non-commuting. In refs. [19, 20] we have explicitly shown this non-commutativity for a particular class of the models in question. Now we briefly describe this example.

4. Harmonic Yang-Mills sigma models. Here we present a particular solution to the constraints (28)-(31). We believe that it shares many features of the general solution still to be found.

It is given by the following ansatz

$$\begin{aligned} h^{1,3N} &= h^{3,1N} = 0; \quad h^{2,2} = h^{2,2}(t, u, v), \quad t^{2,2} = q^{1,1M} q^{1,1M}; \\ h^{2,2[N,M]} &= b^{1,1} f^{NML} q^{1,1L}, \quad b^{1,1} = b^{ia} u_i^1 v_a^1, \quad b^{ia} = \text{const}, \end{aligned} \quad (37)$$

where the real constants f^{NML} are totally antisymmetric. The constraints (28) - (30) are identically satisfied with this ansatz, while (31) is now none other than the Jacobi identity which tells us that the constants f^{NML} are structure constants of some real semi-simple Lie algebra (the minimal possibility is $n = 3$, the corresponding algebra being $so(3)$). Thus the (4, 4) sigma models associated with the above solution can be interpreted as a kind of Yang-Mills theories in the harmonic superspace. They provide the direct non-abelian generalization of the twisted multiplet sigma models with the action (21) which are thus analogs of two-dimensional abelian gauge theory. The action (33) specialized to the case (37) is as follows

$$\begin{aligned} S_{q,\omega}^{YM} &= \int \mu^{-2,-2} \{ q^{1,1M} (D^{0,2} \omega^{1,-1M} + D^{2,0} \omega^{-1,1M} + b^{1,1} \omega^{-1,1L} \omega^{1,-1N} f^{LNM}) \\ &\quad + h^{2,2}(q, u, v) \}. \end{aligned} \quad (38)$$

It is a clear analog of the Yang-Mills action in the first order formalism, $q^{1,1N}$ being an analog of the YM field strength and $b^{1,1}$ of the YM coupling constant.

An interesting specific feature of this “harmonic Yang-Mills theory” is that the “coupling constant” $b^{1,1}$ is doubly charged (this is necessary for the balance of harmonic $U(1)$

charges). Since $b^{1,1} = b^{ia} u_i^1 v_a^1$, we conclude that in the geometry of the considered class of $(4, 4)$ sigma models a very essential role is played by the quartet constant b^{ia} . When $b^{ia} \rightarrow 0$, the non-abelian structure contracts into the abelian one and we reproduce the twisted multiplet action (21). It also turns out that b^{ia} measures the “strength” of non-commutativity of the left and right complex structures on the bosonic target.

In the simplest case of $h^{2,2} = q^{1,1 M} q^{1,1 M}$ the physical bosons part of the action (38) (after fixing a WZ gauge with respect to (34) and a partial elimination of auxiliary fields) is given by the following expression

$$S_{bos} = \int d^2x [dv] \left(\frac{i}{2} g^{0,-1 i M}(x, v) \partial_{--} q_i^{0,1 M}(x, v) \right). \quad (39)$$

Here the fields g and q are subjected to the harmonic differential equations

$$\begin{aligned} \partial^{0,2} g^{0,-1 i M} - 2(b^{ka} v_a^1) f^{MNL} q^{0,1 i N} g_k^{0,-1 L} &= 4i \partial_{++} q^{0,1 i M} \\ \partial^{0,2} q^{0,1 i M} - 2f^{MLN} (b^{ka} v_a^1) q_k^{0,1 L} q^{0,1 i N} &= 0 \end{aligned} \quad (40)$$

and $q^{0,1 i M}$ contains the physical bosonic field as the first component in its v decomposition

$$q^{0,1 i M}(x, v) = q^{ia M}(x) v_a^1 + \dots$$

Solving eqs. (40), one may express the involved functions in terms of $q^{ia M}(x)$, do the v integration in (39) and find the explicit expressions for the bosonic metric and torsion potential. In [19, 20] this was done in the first non-vanishing order in $q^{ia M}$ and b^{ia} . This approximation proved to be sufficient for computing the relevant left and right complex structures (to the same order in fields), checking their properties and finding their mutual commutator. The latter was found to be non-vanishing and proportional to the coupling constant b^{ia} .

Thus in the present case in the bosonic sector we encounter a more general geometry compared to the one associated with twisted $(4, 4)$ multiplets. The basic characteristic feature of this geometry is the non-commutativity of the left and right complex structures. It is easy to check this property also for general potentials $h^{2,2}(q, u, v)$ in (38). It seems obvious that the general case (33), (28) - (31) reveals the same feature. Stress once more that this important property is related in a puzzling way to the non-abelian structure of the analytic superspace actions (38), (33): the coupling constant $b^{1,1}$ (or the Poisson potential $h^{2,2 [M,N]}$ in the general case) measures the strength of the non-commutativity of complex structures.

5. Outlook. The obvious problems for further study of the presented new class of $(4, 4)$ sigma models are to compute the relevant metrics and torsions in a closed form and to try to utilize the corresponding manifolds as backgrounds for some superstrings. An interesting question is as to whether the constraints (28) - (31) admit solutions corresponding to the $(4, 4)$ supersymmetric group manifold WZNW sigma models. The list of appropriate group manifolds has been given in [9]. The lowest dimension manifold with non-commuting left and right structures [11] is that of $SU(3)$. Its dimension 8 coincides with the minimal bosonic manifold dimension at which a non-trivial $h^{2,2 [M,N]}$ in (33) can appear.

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